Averaging techniques for the control of bilinear quantum systems with mixed spectrum

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Structure of the talk

Plan

Physical context

A quantum system evolving in Ω, a finite dimensional Riemannian manifold, is described by its *wave function* ψ in the unit sphere of $L^2(\Omega, \mathbb{C})$. The system is in the subset ω with probability $\int_\omega |\psi|^2 \mathrm{d}\mu$. The time evolution is given by the Schrödinger equation

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When submitted to an external field (*e.g.*, a laser) with time variable intensity, ψ satisfies

$$\mathrm{i}\frac{\partial \psi}{\partial t} = (-\Delta + V(x))\psi(x,t) + u(t)W(x)\psi(x,t)$$

Abstract framework

$$\frac{d\psi}{dt} = A\psi + u(t)B\psi$$

- $\mathcal{H}$: complex Hilbert space with Hilbert product $\langle \cdot, \cdot \rangle$.
- $A : D(A) \to \mathcal{H}$ skew-adjoint ;
- $\{0, 1\} \subset U \subset \mathbb{R}$
- $B : D(B) \to \mathcal{H}$ skew-symmetric such that $D(A) \cap D(B)$ is dense and $A + uB$ is essentially skew-adjoint for every u in U ;

Solution with piecewise constant controls

If $u : [0, T] \to U$ is piecewise constant, $u = \sum_{j=1}^{n} u_j 1_{(t_j, t_j+\tau_j)}$
$\Upsilon^u_{t,0}\psi_0 = e^{(t-t_l)(A+u_l B)} \circ e^{\tau_{l-1}(A+u_{l-1}B)} \circ \cdots \circ e^{\tau_1(A+u_1 B)}\psi_0$

These are *ultra-weak solutions* in the sense of Lions 1958.

Larger class of admissible controls (Kato, 1953)

A-boundedness

B is A bounded if there exists $c_1 > 0$, $c_2 > 0$ such that for every x in $D(A)$, $\|Bx\| \leq c_1\|Ax\| + c_2\|x\|$.

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Well posedness of the bilinear evolution equation $x' = Ax + uBx$

If A is skew-adjoint and B is A-bounded, then Υ^u admits a continuous extension to the set of controls u with bounded variation and taking value in $(-c_1^{-1}, c_1^{-1})$.

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Energy estimate

In this case, $D(A)$ is stable by Υ^u and there exists $C(A,B)$ s.t. $\forall u$,

$$\|A\Upsilon_t^u\psi_0\| \leq C(A,B)e^{C(A,B)TV(u)}(\|A\psi_0\| + \|\psi_0\|)$$

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$TV(u) = |u(0)| + \int_{\mathbb{R}} |u'|\mathrm{d}\lambda$, valid for every $t > 0$.

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- **Linear test**
  - Beauchard, Coron, Laurent, Morancey, Marbach, Nersessyan, Duca,..
  - Find a suitable (Banach, Hilbert,...) subspace $\mathcal{G}$ of $\mathcal{H}$;
  - Prove that the input-output mapping $\Upsilon_t : \mathcal{U} \to \mathcal{H}$ is differentiable (in a suitable sense) ;
  - Prove that $\mathrm{d}\Upsilon_t : \mathcal{U} \to \mathcal{G}$ is a bijection ;
  - Apply some inverse/implicit function theorem.

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- **Linear test**
- **Lyapunov techniques**
  - Nersessyan, Mirrahimi, Rouchon, Beauchard,...
  - Compute $\frac{d}{dt}\|\psi_{target} - x(t)\|$ in terms of u ;
  - Chose u such that $\frac{d}{dt}\|\psi_{target} - x(t)\| < 0$;
  - Conclude with some extension of Laplace invariance theorem ;

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Difficulties

- These techniques heavily rely on the regularity of A and B.
- B must be bounded.

Spectral theorem for self-adjoint operator (dim $\mathcal{H} < \infty$)

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Spectral theorem

The same construction can be extended to unbounded self-adjoint operators in infinite dimensional spaces.

Spectral measure

Projection valued measure

Family $(P_\Omega)_{\Omega \in Leb.me(\mathbb{R})}$ such that

- Each $P_\Omega : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection
- $P_\emptyset = 0$, $P_\mathbb{R} = Id$
- $\Omega = \cup_n \Omega_n$ (disjoint sets), then $\forall \psi$, $P_\Omega \psi = \lim_N \sum_{n=1}^{N} P_{\Omega_n} \psi$
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Measure

For every ψ in $\mathcal{H}$, $\langle \psi, P_\Omega \psi \rangle$ is a Borel measure on $\mathbb{R}$.
By polarization, $\langle \psi, P_\Omega \phi \rangle$ is a complex measure on $\mathbb{R}$.

Spectral theorem for self-adjoint operator

(see for instance Volume 1 of Reed–Simon, Theorem VIII.6)

Theorem

There is a one-to-one correspondence between self-adjoint operators A and projection-valued measure $\{P_\Omega\}$ on $\mathcal{H}$, given by $A = \int_{\mathbb{R}} \lambda dP_\lambda$.

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Functional calculus

$$\langle \psi, g(A)\phi \rangle = \int_{-\infty}^{+\infty} g(\lambda)\mathrm{d}(\psi, P_\lambda \phi)$$

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The measure P_λ may be decomposed in three parts

1. "discrete" part (atoms), gives the eigenvalues and eigenvectors.
2. absolutely continuous (L^1 density) part with respect to Lebesgue measure.
3. singularly continuous part.

Motivation of this talk

Orthogonal decomposition

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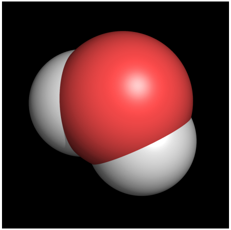
- Most of the results in the literature deal with the case $\mathcal{H}_c = \{0\}$.
- Some exceptions
  - Mirrahimi 2008 : Lyapunov techniques, dimension 1, B bounded.
  - TC 2012-2013 : averaging, B bounded and continuous spectrum split from pointwise spectrum.
- No available result about eigenvalues immersed in the continuous spectrum.

Example of the vibrations of OH-bonds in a water molecule

Morse potential and Mecke interaction potential



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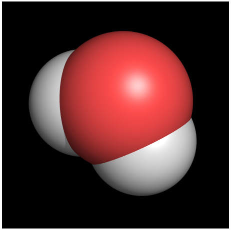

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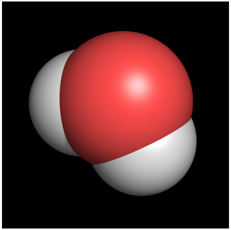

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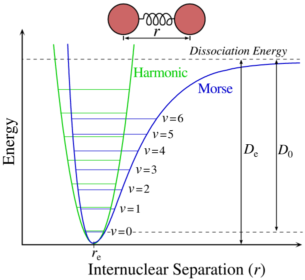
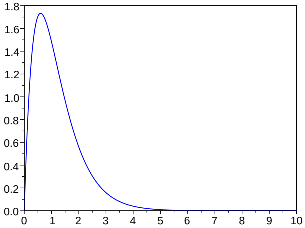
$$\mathrm{i}\frac{\partial \psi}{\partial t} = \Big(\Delta \underbrace{- D_e[1 - e^{-\beta(r - r_e)}]^2}_{\text{Morse potential}}\Big)\psi + u(t)\mu_0 r e^{-\frac{r}{r_e}}\psi$$





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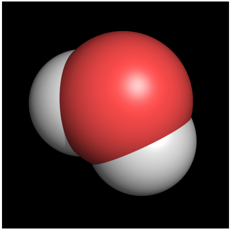

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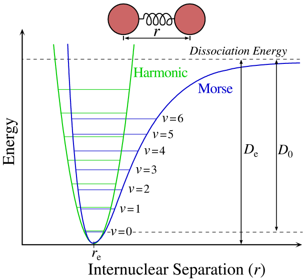

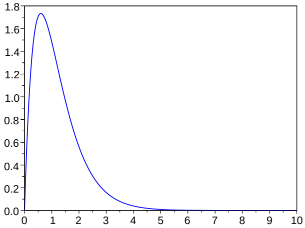

Plan

Averaging techniques (principle)

Newton, Lagrange, Poincaré, Bogoliubov, Verduyn-Lunel, Hale, Fox,..

$$\left\{\begin{array}{lll} \dot{x} & = & \varepsilon f(t,x) \\ x(0) & = & x_0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lll} \dot{y} & = & \varepsilon \bar{f}(y) \\ y(0) & = & x_0 \end{array}\right.$$

have solutions "ε-close" for $t \leq \frac{1}{\varepsilon}$ if f is "regular" and

$$\bar{f}(y) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t,y)\mathrm{d}t.$$

Well known in finite dimension with C^2 functions.

cf Verlhust many extensions in the last century.

Verduyn-Lunet et Hale, 1990 extension in infinite dimension for f C^1.

Quantum averaging

Plugging $y = e^{-tA}x$ in $\dot{x} = (A + uB)x$ gives $\dot{y} = u(t)e^{-tA}Be^{tA}y$. If A is diagonal (eigenvalues $\mathrm{i}\lambda_1, \mathrm{i}\lambda_2, \ldots..$), using $\varepsilon u(t)$

$$\dot{z} = \varepsilon \left(\quad b_{jk} \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{\mathrm{i}(\lambda_j - \lambda_k)t} u(t)\mathrm{d}t \quad \right) z$$

Periodic excitations : principle

If u is $\frac{2\pi}{\lambda_k - \lambda_j}$-periodic, there remains only the transition $j \leftrightarrow k$.

First results of averaging

U. Boscain, M. Caponigro, TC, P. Mason, M. Sigalotti, 2009–2012

Assumption : basis of eigenvectors

1. A is skew-adjoint with domain $D(A)$;
2. B is skew-symmetric ;
3. there exists a basis of $\mathcal{H}$ made of eigenvectors of A ;
4. ϕ_1 and ϕ_2 are two eigenvectors of A of norm 1, associated with eigenvalues $-\mathrm{i}\lambda_1$ and $-\mathrm{i}\lambda_2$;
5. $\lambda_1 < \lambda_2$ and $\langle \phi_1, B\phi_2 \rangle \neq 0$;
6. for every eigenvalues $-\mathrm{i}\mu$ and $-\mathrm{i}\mu'$ of A, associated with eigenvectors v and v', $|\lambda_1 - \lambda_2| = |\mu - \mu'|$ implies $\{\lambda_1, \lambda_2\} = \{\mu, \mu'\}$ or $\{\lambda_1, \lambda_2\} \cap \{\mu, \mu'\} = \emptyset$ or $\langle v, Bv' \rangle = 0$.

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Proposition

Let $(A, B, \phi_1, \phi_2, \lambda_1, \lambda_2)$ satisfy Assumption 1. Then there exists $u_\varepsilon : [0, T_\varepsilon] \to (-\frac{1}{a}, \frac{1}{a})$ such that $\|\Upsilon^{u_\varepsilon}_{T_\varepsilon}\phi_1 - \phi_2\| < \varepsilon$.

- Explicit form of the control : piecewise constant approximation of $t \mapsto \sin(|\lambda_1 - \lambda_2|t)/n$ for n large enough.
- Robustness with respect to perturbations of the spectrum of A (as long as λ_1 and λ_2 remain unchanged).
- Robustness with respect to perturbations of B (as long as $\langle\phi_1, B\phi_2\rangle$ does not vanish).
- In general, no explicit estimates of n (and the time) in terms of A, B and ε.

Slightly better averaging results

Assumption : mixed spectrum + possibly unbounded coupling term

1. A is skew-adjoint with domain $D(A)$;
2. B is skew-symmetric ;
3. B is bounded relatively to A : $\forall \psi$ in $D(A)$, $\|B\psi\| < a\|A\psi\|$;
4. ϕ_1 and ϕ_2 are two eigenvectors of A of norm 1, associated with eigenvalues $-\mathrm{i}\lambda_1$ and $-\mathrm{i}\lambda_2$;
5. $\lambda_1 < \lambda_2$ and $\langle \phi_1, B\phi_2 \rangle \neq 0$;
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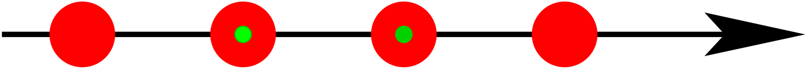
7. the essential spectrum of $\mathrm{i}A$ (if any) does not accumulate in any of these four points : $2\lambda_1 - \lambda_2, \lambda_1, \lambda_2$ and $2\lambda_2 - \lambda_1$.



Slightly better averaging results

Proposition

Let $(A, B, \phi_1, \phi_2, \lambda_1, \lambda_2)$ satisfy Assumption 1. Then, for every $\varepsilon > 0$, for every $r < 1$, there exists $u_\varepsilon : [0, T_\varepsilon] \to (-\frac{1}{a}, \frac{1}{a})$ such that $\||A|^r(\Upsilon^{u_\varepsilon}_{T_\varepsilon}\phi_1 - \phi_2)\| < \varepsilon$.

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- Robustness with respect to perturbations of the spectrum of A (as long as λ_1 and λ_2 remain unchanged).
- Robustness with respect to perturbations of B (as long as $\langle\phi_1, B\phi_2\rangle$ does not vanish).
- Explicit estimates of n (and the time) in terms of A, B and ε.
- For most of the usual systems, convergence in higher norms can be recovered from regularity arguments.

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Problem

If B is not bounded with respect to A, I do not know how to estimate the time.

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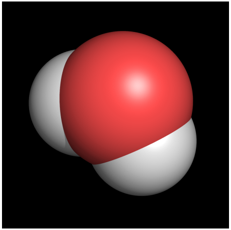

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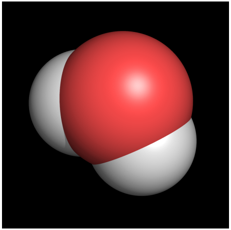

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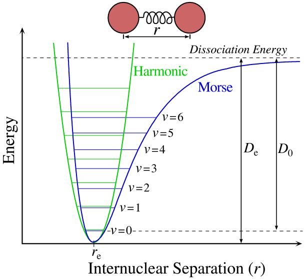

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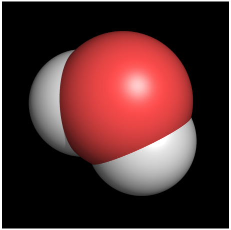

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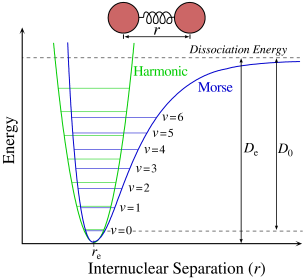

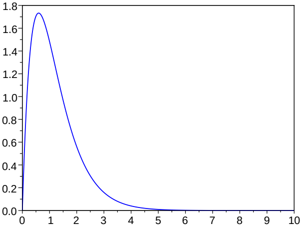

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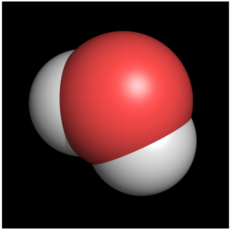

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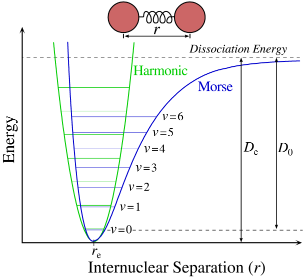
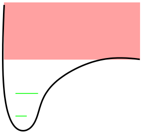
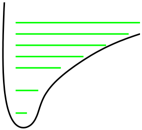
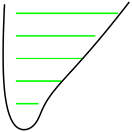




Morse potential



Morse $+\frac{1}{10}x$



Morse $+x$

A misleading result

There exist a sequence ω_n converging to $\lambda_2 - \lambda_1$ and a sequence u_n of $\frac{2\pi}{\omega_n}$-periodic piecewise constant functions such that

$$\|\Upsilon^{\frac{u_n}{n}}_{n,0}\phi_1 - \phi_2\| \to 0.$$

Plan

1. Bilinear quantum systems
   - Motivation and framework
2. Averaging techniques
   - Periodic excitations and non degenerate transitions
3. A decoupling result
   - Variation on the RAGE theorem
4. Conclusion
   - Achievements and some open questions

The RAGE theorem

(see Reed–Simon, Volume 3, Appendix XI.17)

Definition

C is A-compact if $C(A+1)^{-1}$ is compact ($\Leftrightarrow$ C sends bounded sets of $D(A)$ to relatively compact sets in $\mathcal{H}$).

The RAGE theorem

(see Reed–Simon, Volume 3, Appendix XI.17)

Definition

C is A-compact if $C(A+1)^{-1}$ is compact ($\Leftrightarrow$ C sends bounded sets of $D(A)$ to relatively compact sets in $\mathcal{H}$).

RAGE Theorem (Ruelle, Amrein, Georgescu, Enss)

If A is skew-adjoint, and C is bounded and A-compact, then for every ψ in $\pi^c\mathcal{H}$,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|Ce^{tA}\psi\| \mathrm{d}t = 0$$

A time-varying RAGE theorem

Proposition (N. Boussaïd, M. Caponigro, TC)

Let A be a skew-adjoint operator on $\mathcal{H}$. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^1(\mathbb{R})$ such that $(\widehat{|u_n|})_{n\in\mathbb{N}}$ converges pointwise to 0 on $\mathbb{R}\setminus D$ with $D\subset\mathbb{R}$ countable. Then, for any compact operator C on $\mathcal{H}$, for any $\psi\in\mathcal{H}$

$$\lim_{n\to\infty}\sup_{f\in B(\mathbb{R}),\,\|f\|_\infty\leq 1}\int_{\mathbb{R}}|u_n(t)|\left\|Cf(iA)e^{tA}\pi_c\psi\right\|\,\mathrm{d}t=0.$$

A time-varying RAGE theorem : sketch of the proof

$$\lim_{n\to\infty}\int_{\mathbb{R}} |u_n(t)| \left\| Ce^{tA}\pi_c\psi \right\| \mathrm{d}t = 0.$$

- Assume C has rank one
- By Cauchy-Schwarz : it is enough to show

$$\lim_{n\to\infty}\int_{\mathbb{R}} |u_n(t)| \left\| Ce^{tA}\pi_c\psi \right\|^2 \mathrm{d}t = 0.$$

- $\|Ce^{tA}\psi\|^2 = |\langle\phi, e^{tA}\psi\rangle|^2 = |\int_{\mathbb{R}} e^{ti\lambda}\langle\phi, dP_\lambda(\psi)\rangle|^2$ the integral can be rewritten in term of Fourier transform of the spectral measure
- Wiener theorem $\int_{\mathbb{R}} |u_n||\hat{\mu}|^2 \mathrm{d}t = \int_{\mathbb{R}} \overline{\widehat{|u_n|}(-\cdot) * \mu} \, \mathrm{d}\mu \to \sum_{x\in\mathbb{R}, x_0\in D} \overline{w(x_0)\mu(\{x+x_0\})}\mu(\{x\})$

A new result

Decoupling theorem (N. Boussaïd, M. Caponigro, TC)

Assumptions

- A skew-adjoint, B is A-compact
- $\Upsilon^{u,p}$: propagator associated with $\pi_p A \pi_p$ and $\pi_p B \pi_p$
- (u_n) is bounded in L^1, bounded in norm TV, and $\widehat{|u_n|}$ tends to zero on $\mathbb{R} \setminus D$ with D countable.

Conclusion

$$\lim_{n\to\infty} \sup_{t>0} \|\pi_p \Upsilon_t^{u_n} \phi_0 - \Upsilon_t^{u_n,p} \pi_p \phi_0\| = 0.$$

A decoupling result, proof 1/4

Assumption : B is bounded with finite rank. By Duhamel formulas (Kato, 1953, strong solutions if $\phi_0 \in D(A)$)

$$\pi_p \Upsilon_t^{u_n} \phi_0 = e^{tA} \pi_p \phi_0 + \int_0^t e^{(t-s)A} u_n(s) \pi_p B \pi_p \Upsilon_s^{u_n} \phi_0 \,\mathrm{d}s + \int_0^t e^{(t-s)A} u_n(s) \pi_p B \pi_c \Upsilon_s^{u_n} \phi_0 \,\mathrm{d}s$$

$$\pi_p(A) \Upsilon_t^{u_n,p} \phi_0 = e^{tA} \pi_p \phi_0 + \int_0^t e^{(t-s)A} u_n(s) \pi_p B \pi_p \Upsilon_s^{u_n,p} \phi_0 \,\mathrm{d}s.$$

$$\begin{aligned} &\left\| \pi_p \Upsilon_t^{u_n} \phi_0 - \Upsilon_t^{u_n,p} \pi_p \phi_0 \right\| \\ &\leq \left\| \int_0^t e^{(t-s)A} u_n(s) \pi_p B \pi_c \Upsilon_s^{u_n} \phi_0 \,\mathrm{d}s \right\| \\ &+ \int_0^t |u_n(s)| \|B\| \left\| \pi_p \Upsilon_s^{u_n} \phi_0 - \Upsilon_s^{u_n,p} \pi_p \phi_0 \right\| \,\mathrm{d}s. \end{aligned}$$

A decoupling result, proof 2/4

Gronwall Lemma :

$$\left\|\pi_p \Upsilon_t^{u_n}\phi_0 - \Upsilon_t^{u_n,p}\pi_p\phi_0\right\| \leq \left\|\int_0^t e^{(t-s)A}u_n(s)\pi_p B\pi_c \Upsilon_s^{u_n}\phi_0 \,\mathrm{d}s\right\| e^{\|u_n\|_{L^1(\mathbb{R})}\|B\|}.$$

Hence the result amounts to prove that

$$\lim_{n\to\infty}\sup_{t>0}\left\|\int_0^t e^{(t-s)A}u_n(s)\pi_p B\pi_c \Upsilon_s^{u_n}\phi_0 \,\mathrm{d}s\right\| = 0,$$

or

$$\lim_{n\to\infty}\sup_{t>0}\left\|\int_0^t e^{-sA}u_n(s)\pi_p B\pi_c \Upsilon_s^{u_n}\phi_0 \,\mathrm{d}s\right\| = 0.$$

A decoupling result, proof 3/4

$$e^{-sA}\Upsilon_s^{u_n}\phi_0 = \phi_0 + \int_0^s e^{-\tau A} u_n(\tau) B \Upsilon_\tau^{u_n}\phi_0 \, \mathrm{d}\tau.$$

The set

$$\mathcal{T}((u_n)_{n\in\mathbb{N}}) := \{\Upsilon_\tau^{u_n}\phi_0, \ \tau \geq 0, \ n \in \mathbb{N}\}$$

is bounded in $D(A)$ hence (since B is A-compact)

$$B\mathcal{T}((u_n)_{n\in\mathbb{N}}) := \{B\Upsilon_\tau^{u_n}\phi_0, \ \tau \geq 0, \ n \in \mathbb{N}\}$$

is relatively compact in $\mathcal{H}$. Hence, for every $\delta > 0$, there exist $\psi_1,\ldots,\psi_N$ in $\mathcal{H}$ such that

$$B\mathcal{T}((u_n)_{n\in\mathbb{N}}) \subset \bigcup_{k=1}^{N} B_{\mathcal{H}}(\psi_k, \delta).$$

Partition of the unity : $\sum_i^N F_i = 1$, $0 \leq F_i \leq 1$, $F_i(x) = 0$ if $\|x - \psi_i\| > 2\delta$.

A decoupling result, proof 4/4

$$\int_0^s e^{-\tau A} u_n(\tau) B \Upsilon_\tau^{u_n} \phi_0 \,\mathrm{d}\tau = \int_0^s e^{-\tau A} u_n(\tau) B \Big(\sum_{i=1}^N F_i(\Upsilon_\tau^{u_n}\phi_0)\Big) \Upsilon_\tau^{u_n} \phi_0 \,\mathrm{d}\tau$$

$$= \int_0^s \sum_{i=1}^N F_i(\Upsilon_\tau^{u_n}\phi_0) e^{-\tau A} u_n(\tau) B \Upsilon_\tau^{u_n} \phi_0 \,\mathrm{d}\tau$$

$$\approx \int_0^s \sum_{i=1}^N F_i(\Upsilon_\tau^{u_n}\phi_0) e^{-\tau A} u_n(\tau) B \psi_i \,\mathrm{d}\tau$$

$$+ \left(\leq \|B\| \|u_n\|_{L^1} \delta\right)$$

A decoupling result, proof 4/4

$$\int_0^s e^{-\tau A} u_n(\tau) B \Upsilon^{u_n}_\tau \phi_0 \, \mathrm{d}\tau = \int_0^s e^{-\tau A} u_n(\tau) B \Big(\sum_{i=1}^N F_i(\Upsilon^{u_n}_\tau \phi_0)\Big) \Upsilon^{u_n}_\tau \phi_0 \, \mathrm{d}\tau$$

$$= \int_0^s \sum_{i=1}^N F_i(\Upsilon^{u_n}_\tau \phi_0) e^{-\tau A} u_n(\tau) B \Upsilon^{u_n}_\tau \phi_0 \, \mathrm{d}\tau$$

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$$+ (\leq \|B\| \|u_n\|_{L^1} \delta)$$

- Use the time varying version of RAGE (proof ok if B is bounded).

A decoupling result, proof 4/4

$$\int_0^s e^{-\tau A} u_n(\tau) B \Upsilon_\tau^{u_n} \phi_0 \,\mathrm{d}\tau = \int_0^s e^{-\tau A} u_n(\tau) B \Big(\sum_{i=1}^N F_i(\Upsilon_\tau^{u_n}\phi_0)\Big) \Upsilon_\tau^{u_n} \phi_0 \,\mathrm{d}\tau$$

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- Use the time varying version of RAGE (proof ok if B is bounded).
- If B is not bounded :

A decoupling result, proof 4/4

$$\int_0^s e^{-\tau A} u_n(\tau) B \Upsilon_\tau^{u_n}\phi_0 \,\mathrm{d}\tau = \int_0^s e^{-\tau A} u_n(\tau) B \Big(\sum_{i=1}^N F_i(\Upsilon_\tau^{u_n}\phi_0)\Big) \Upsilon_\tau^{u_n}\phi_0 \,\mathrm{d}\tau$$

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$$\approx \int_0^s \sum_{i=1}^N F_i(\Upsilon_\tau^{u_n}\phi_0) e^{-\tau A} u_n(\tau) B \psi_i \,\mathrm{d}\tau + (\leq \|B\| \|u_n\|_{L^1} \delta)$$

- Use the time varying version of RAGE (proof ok if B is bounded).
- If B is not bounded :
  - since B is A-compact, $B = B_f + B_c$ with B_f having finite rank and $B_c(A+1)^{-1}$ arbitrary small

A decoupling result, proof 4/4

$$\int_0^s e^{-\tau A} u_n(\tau) B \Upsilon_\tau^{u_n} \phi_0 \,\mathrm{d}\tau = \int_0^s e^{-\tau A} u_n(\tau) B \Big(\sum_{i=1}^N F_i(\Upsilon_\tau^{u_n}\phi_0)\Big) \Upsilon_\tau^{u_n} \phi_0 \,\mathrm{d}\tau$$

$$= \int_0^s \sum_{i=1}^N F_i(\Upsilon_\tau^{u_n}\phi_0) e^{-\tau A} u_n(\tau) B \Upsilon_\tau^{u_n} \phi_0 \,\mathrm{d}\tau$$

$$\approx \int_0^s \sum_{i=1}^N F_i(\Upsilon_\tau^{u_n}\phi_0) e^{-\tau A} u_n(\tau) B \psi_i \,\mathrm{d}\tau$$

$$+ (\leq \|B\| \|u_n\|_{L^1} \delta)$$

- Use the time varying version of RAGE (proof ok if B is bounded).
- If B is not bounded :
  - since B is A-compact, $B = B_f + B_c$ with B_f having finite rank and $B_c(A+1)^{-1}$ arbitrary small
  - small perturbation on B induces small perturbations on $\Upsilon_t^u \psi$

Plan

Take home message

Averaging

- Well-known control technique.
- Works very well on the pure-point spectrum part with very few regularity assumptions.
- Works also with mixed spectrum if B is A-compact.

Using compactness to deal with continuous spectrum

- If B is A-compact, averaging controls do not couple continuous and pure point spectrum.
- Everything is as if the continuous part did not exist.

Some open questions

- Should work when B is A-bounded and $\pi_p B \pi_c$ is A compact ?
- Robustness of the controls ?
- How much time do you need for this ?

Some open questions

- Should work when B is A-bounded and $\pi_p B \pi_c$ is A compact ?
- Robustness of the controls ?
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Thank you.

Acknowledgment

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